Table 1 Euler beam elastic mode shape parameters

Boundary condition[a]	β_n	θ	A	B
SS-SS	$n\pi$	0	0	0
CL-FR	$(n - \frac{1}{2})\pi$	$-\pi/4$	1	$(-1)^{n+1}$
CL-CL	$(n + \frac{1}{2})\pi$	$-\pi/4$	1	$(-1)^{n+1}$
FR-FR	$(n + \frac{1}{2})\pi$	$+3\pi/4$	1	$(-1)^{n+1}$
SS-CL	$(n + \frac{1}{4})\pi$	0	0	$(-1)^{n+1}$
SS-FR	$(n + \frac{1}{4})\pi$	0	0	$(-1)^{n}$

[a]SS = simply supported, CL = clamped, and FR = free.

In the above, EI is the bending stiffness, m is the mass per unit length, ℓ is the beam length, and x is the beam coordinate, nondimensionalized with respect to length ℓ, and varying between 0 and 1. The mode shape $\phi_n(x)$ satisfies the condition

$$\int_0^1 \phi_n^2(x)\,dx = 1 \tag{5}$$

Following a suggestion in Ref. 3, Eq. (1) can be rearranged, by adding and subtracting $\sinh\beta_n x$, into the form

$$\phi_n(x) = \cos\beta_n x - \alpha_n \sin\beta_n x + e^{-\beta_n x} + (1-\alpha_n)\sinh\beta_n x \tag{6}$$

Introducing Eq. (4), the $(1-\alpha_n)$ term in Eq. (6) becomes

$$1-\alpha_n = \frac{\cos\beta_n - \sin\beta_n - e^{-\beta_n}}{\sinh\beta_n - \sin\beta_n} \tag{7}$$

The solution of Eq. (3) for β_n is found by plotting the right- and left-hand sides and is given, for $n \geq 2$, to less than 0.01% accuracy as

$$\beta_n \approx (n + \tfrac{1}{2})\pi \tag{8}$$

For $n = 2$, one has $\beta_2 = 7.8540$, $\sinh\beta_2 = 1288.0$, $\exp(-\beta_2) = 0.000388$. Under these conditions and using Eq. (8), the relation Eq. (7) can be approximated as

$$(1-\alpha_n) \approx \frac{-(-1)^n}{\sinh\beta_n} \approx (-1)^{n+1}\,2e^{-\beta_n} \tag{9}$$

The last two terms of Eq. (6) then can be combined to give

$$e^{-\beta x}[1-(-1)^{n+1}e^{-\beta_n}] + (-1)^{n+1}e^{-\beta_n(1-x)} \tag{10}$$

Since $\exp(-\beta_n) << 1$ and from Eq. (4) one has $\alpha_n \approx 1$, the mode shape given by Eq. (6) can be rewritten in the simple form as

$$\phi_n = \cos\beta_n x - \sin\beta_n x + e^{-\beta_n x} + (-1)^{n+1}e^{-\beta_n(1-x)} \tag{11}$$

Equation (11) represents a shifted sinusoidal wave with small exponential end corrections. It is easily evaluated numerically, nodes and peaks are readily identified, and it is valid for any mode $n \geq 2$. Even for $n = 1$, it gives only a 0.7% error in amplitude at the midpoint.

Similar expressions can be developed for other boundary conditions. The higher vibration modes of uniform Euler beams for $n \geq 2$ can then be simply expressed as

$$\phi_n(x) = \sqrt{2}\sin(\beta_n x + \theta) + Ae^{-\beta_n x} + Be^{-\beta_n(1-x)} \tag{12}$$

where the constants β_n, θ, A, and B are given in Table 1 for some common boundary conditions. The corresponding frequencies are given by Eq. (2). All modes are normalized such that Eq. (5) holds. These modes also apply for $n = 1$ with less than a 1% error, except for the clamped-free case. Not only are these modes easily evaluated numerically, but they are immediately identified as simple shifted sinusoidal waves with small, confined, exponential edge-zone corrections to match the required boundary conditions at each end. In terms of the mode wavelength $\lambda = 2\pi/\beta_n$, the edge-zone correction dies out to 5% of its maximum end value (i.e., $\beta_n x = 3$) in roughly half a wavelength. The forms of Eq. (12) can be very useful in performing large multimode Rayleigh-Ritz type analyses for beams, plates, and shells with different boundary conditions. They also show clearly that standing waves of a beam have the same sinusoidal structure as traveling waves except for small edge-zone regions near the ends where reflections and absorptions occur. The analysis developed here may also prove useful in dealing with Timoshenko beams and related shear-type structures. An extensive survey of related work and applications of edge effects has been summarized by Elishakoff.[4] Also, the form of Eq. (11) has been presented earlier by Dowell[5] for the clamped-free and free-free beam cases.

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Active Vibration Isolation by Polymeric Piezoelectret with Variable Feedback Gains

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ACTIVE vibration involves the cancellation of unwanted oscillation with an equal and opposite excitation that is artificially generated by an arrangement of active vibration control devices. Numerous active control systems using electromagnetic,[1] pneumatic,[2] hydraulic,[3] and viscoelastic[4] force generators have been investigated. In recent years, piezoelectrets were also used as active actuators for active vibration control of distributed parameter systems[5-8] because of their potential applications in large flexible and precision space structures.

This paper presents a theoretical and experimental study of an active piezoelectric vibration isolation technique using polymeric piezoelectric polyvinylidence fluoride (PVDF) with variable feedback gains. Injecting feedback voltage into a PVDF isolator results in a thickness change of the isolator due to its converse piezoelectric effect. If this thickness change is adjusted 180 deg out of phase with the base excitation, the PVDF isolator (with appropriate feedback gain) can effectively cancel any disturbance from the base. Theory on active piezoelectric vibration isolation with general mechanical and

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electric boundary conditions is developed. Analytical solutions are compared with experimental results.

Theory Development

In this section, a piezoelectric isolator with general electrical (simulating feedback voltage) and mechanical (simulating base excitation) boundary conditions is investigated. A general solution is to be developed so that active vibration isolation resulting from the converse piezoelectric effect can be evaluated.

The converse piezoelectricity can be expressed as[9]

$$\epsilon_{33} = d_{33} \cdot E + \kappa \cdot \sigma_{33} \tag{1a}$$

$$\sigma_{33} = (1/\kappa)\,[\epsilon_{33} - d_{33} \cdot E] \tag{1b}$$

where ϵ is the strain, σ the stress, E the electric field, d the piezoelectric constant, and κ the elastic compliance. For a PVDF isolator with an effective area A, the equation of motion in transverse vibration can be written

$$\frac{\partial q_3^2}{\partial t^2} = \frac{1}{\rho\kappa}\left(\frac{\partial \epsilon_{33}}{\partial \alpha_3} - d_{33} \cdot \frac{\partial E}{\partial \alpha_3}\right) \tag{2}$$

where q_3 is the transverse displacement in the α_3 direction and ρ the mass density. Assume that the electric field is constant over the thickness, i.e., $(\partial E/\partial \alpha_3) = 0$, and the transverse strain is defined as

$$\epsilon_{33} = \frac{\partial q_3}{\partial \alpha_3} \tag{3}$$

Thus, a one-dimensional wave equation for the piezoelectric PVDF isolator can be derived:

$$\ddot{q}_3 = \lambda^2 \frac{\partial^2 q_3}{\partial \alpha_3^2} \tag{4}$$

where λ is the velocity of wave propagation in the PVDF medium, $\lambda = \sqrt{1/\rho\kappa}$. Applying the separation of variables method and using Eq. (3) yield a general solution of ϵ_{33}:

$$\epsilon_{33}(\alpha_3,t) = (\omega/\lambda)[c_0 \cos(\omega\alpha_3/\lambda) - c_1 \sin(\omega\alpha_3/\lambda)] \cdot (c_2 \sin\omega t + c_3 \cos\omega t) \tag{5}$$

where c_0, c_1, c_2, and c_3 are constants to be determined by boundary conditions. Consider two general electrical boundary conditions and two general mechanical boundary conditions at $\alpha_3 = 0$ and ℓ (ℓ is the thickness of the PVDF isolator):

$$\sigma_{33} = \sigma_{33_0} \sin\omega t, \qquad E = E_0 \sin\omega t \text{ at } \alpha_3 = 0 \tag{6a}$$

$$\sigma_{33} = \sigma_{33_0} \sin\omega t, \qquad E = E_0 \sin\omega t \text{ at } \alpha_3 = \ell \tag{6b}$$

Note that σ_{33_0} and E_0 are induced by base excitation and feedback voltage, respectively, ($E_0 = V_0/\ell$ where V_0 is the feedback voltage) in later analysis. Substituting Eqs. (6) into Eq. (5) and using Eq. (3) yield a steady-state solution of q_3:

$$q_3(t) = \int_0^\ell (d_{33}E_0 + \kappa\sigma_{33_0})\left\{\cos\left[\left(\omega/\lambda\right)\alpha_3\right] + \frac{1-\cos(\Omega)}{\sin(\Omega)}\sin\left[\left(\omega/\lambda\right)\alpha_3\right]\right\} \cdot \sin\omega t \cdot d\alpha_3$$

$$= \ell\,(d_{33}E_0 + \kappa\sigma_{33_0})\frac{\tan(\Omega/2)}{(\Omega/2)}\sin\omega t \tag{7}$$

where $\Omega = (\omega\ell/\lambda)$. The induced acceleration $G_{fb}(t)$ expressed in g (gravity) is generated by the PVDF converse piezoelectric effect, and it can be expressed as

$$G_{fb}(t) = \frac{1}{g}\ddot{q}_3(t) = -\frac{\omega^2\ell}{g}(d_{33}E_0 + \kappa\sigma_{33_0}) \cdot \sin\omega t\,\frac{\tan(\Omega/2)}{(\Omega/2)} \tag{8}$$

It is intended that this converse effect be used to actively isolate a seismic mass m_s from the base excitation. Substituting $E_0 = V_0/\ell$ into Eq. (8) yields a converse piezoelectricity-induced force $F_{fb}(t)$

$$F_{fb}(t) = m_s \cdot g \cdot G_{fb}(t) = -m_s \cdot \omega^2 \cdot (d_{33}V_0 + \kappa\ell\,\sigma_{33_0}) \cdot \sin\omega t\,\frac{\tan(\Omega/2)}{(\Omega/2)} \tag{9}$$

Similarly, the force $F_b(t)$ introduced by the base excitation is given by

$$F_b(t) = m_\tau \cdot g \cdot G_b(t) \tag{10}$$

where m_τ is the total mass (including m_s) on a shaker. The resultant acceleration G_r due to the combining effects of excitations and feedbacks can be obtained by balancing the forces,

$$G_r = \frac{1}{g}\frac{F_b + F_{fb}}{m_\tau} = \left[G_{b_0} - \omega^2 \cdot \left(\frac{d_{33}}{g} \cdot V_{fb_0} + \kappa\ell \cdot G_{b_0}\frac{m_s}{A}\right) \times \frac{\tan(\Omega/2)}{(\Omega/2)}\frac{m_s}{m_\tau}\right] \cdot \sin\omega t \tag{11}$$

Substituting all material properties gives $[\tan(\Omega/2)]/(\Omega/2) \cong 1$. Thus,

$$G_{r_0} \cdot \sin\omega t = G_{b_0} - \omega^2 \cdot \left[\frac{d_{33}}{g}V_{fb_0} + \kappa\ell \cdot G_{b_0}\frac{m_s}{A}\right]\frac{m_s}{m_\tau}\sin\omega t \tag{12}$$

The vibration isolation resulting from the feedback-induced converse effect can be defined as the difference between the resultant acceleration and the base excitation. The isolation percentage R can be written as

$$R = \frac{G_b(t) - G_r(t)}{G_b(t)} \times 100 = \frac{\omega^2}{G_{b_0}}\left(\frac{d_{33}}{g}V_{fb_0} + \kappa\ell \cdot G_{b_0}\frac{m_s}{A}\right) \cdot \frac{m_s}{m_\tau} \times 100 \tag{13}$$

The gradient of the isolation surface with respect to the excitation frequency and the feedback voltage is evaluated when the base excitation is constant ($G_{b_0} = G$) and feedback gain varies ($V_{fb_0} = C \times V_0$, where C is the feedback gain and V_0 is the accelerometer output). Thus,

$$R = \frac{\omega^2}{G} \cdot \left[\frac{d_{33}}{g}C \cdot V_0 + G \cdot \kappa\ell\frac{m_s}{A}\right] \cdot \frac{m_s}{m_\tau} \times 100 \tag{14}$$

Substituting all material properties into the equation, it is observed that the second term is small compared with the first term. Thus, at constant feedback voltage, the active piezoelectric isolation is a quadratic funciton of frequency, and at constant frequency it is a linear function of feedback voltage.

Experimentation

A physical model made of PVDF polymer, Fig. 1, was designed and tested to validate the theory discussed earlier. The model base is made of two ¼-in.-thick layers. The bottom layer is steel and is provided with a 10-32 stud so that the model can be mounted on a shaker. The second layer is made of plexiglas. Then, a 1-mm-thick layer of PVDF polymer with an effective surface area of 4×10^{-4} m^2 (2 × 2 cm) is epoxied to the plexiglas layer below. A second layer of plexiglas is glued onto the top surface of the PVDF to provide identical boundary conditions on either side of the PVDF isolator. An interchangeable metal plate is screwed onto this top plexiglas. A miniaccelerometer is attached above this metal plate.

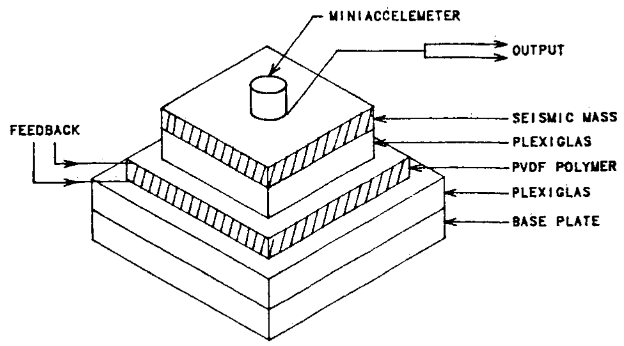

Fig. 1 Physical model with polymeric piezoelectric isolator.

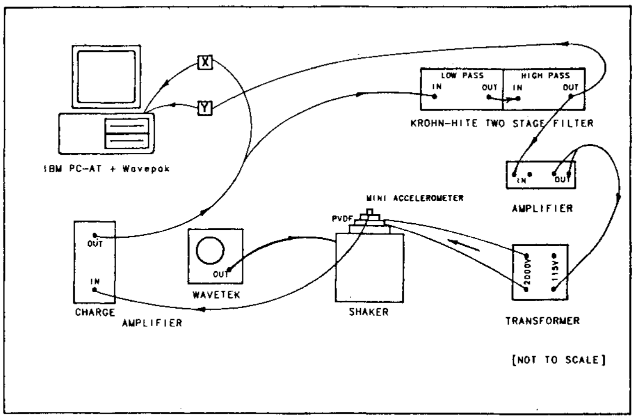

Fig. 2 Experimental setup for the active piezoelectric vibration isolation.

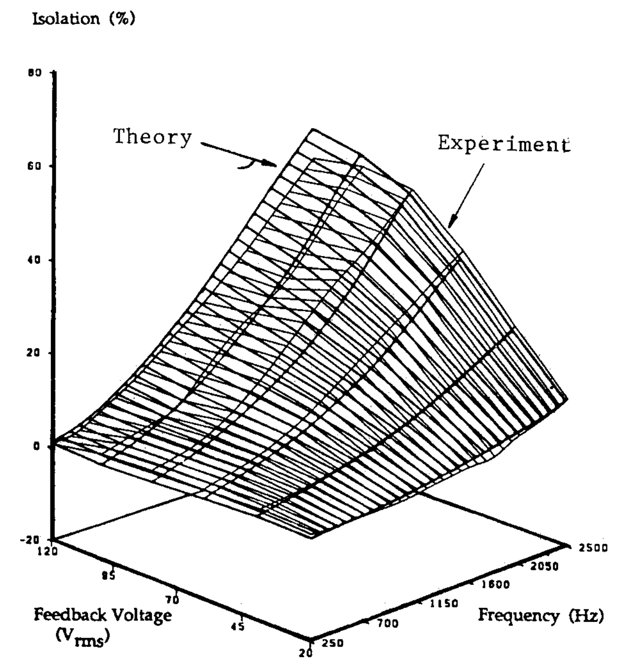

Fig. 3 Active vibration isolation using polymeric piezoelectric PVDF.

The model was mounted on a shaker, which was excited at various frequencies and amplitudes using a Wavetek function generator. The seismic mass acceleration was monitored by the miniaccelerometer. The acceleration signal was then phase-shifted, amplified, and applied across the PVDF layer in such a way that the PVDF vibration was 180 deg out of phase with that of the base. The experimental setup is shown in Fig. 2. The base excitation was kept constant in this set of experiments. Four excitation amplitudes to the shaker were

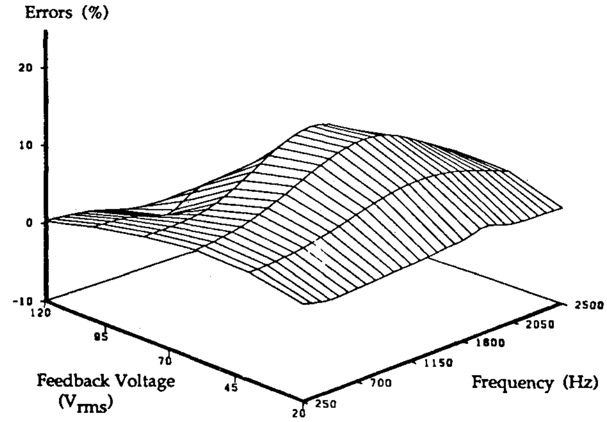

Fig. 4 Absolute percentage errors between theoretical and experimental results.

chosen for each frequency. Since the base excitation was a constant, the feedback gain was varied so that the feedback voltage injected into the PVDF isolator could be controlled.

Results and Discussion

The physical model has a natural frequency of 6000 Hz. Experimental and theoretical results of active piezoelectric vibration isolation are presented in Fig. 3. Experimentally, the active vibration isolation is found to vary between 0.3% at 500 Hz and a feedback voltage of 12 V_{rms} to a maximum of 48% at 2500 Hz and a feedback voltage of 85 V_{rms}. Analytically, the isolation ranges from 0.07% at 250 Hz and a feedback voltage of 11.4 V_{rms} to a maximum of 47.5% at 2500 Hz and feedback of 53.4 V_{rms}. The upper frequency was limited to 25 kHz because of the frequency restriction of the transformer. It is observed that, due to energy dissipation at high feedback voltages, the experimental data are lower than the theoretical predictions. The absolute percentage differences between analytical and experimental values of the resultant seismic mass acceleration are found to be between a low of 0.02% at 500 Hz and 12 V_{rms} feedback to a high of 6.8% at 1900 Hz and 87 V_{rms} feedback (Fig. 4). The possible causes of the errors could be: 1) an interaction of direct and converse effects when both frequency and feedback voltage are low, 2) the feedback signal not being exactly 180 deg out of phase with the seismic mass acceleration, 3) the energy dissipation of the PVDF isolator at high feedback voltages, and 4) the nonlinearity associated with the PVDF physical model. Both the experimental and analytical results show that the isolation gradient is a linear function of excitation voltage at constant frequency and a quadratic function of frequency at constant excitation voltage.

Conclusions

A new active vibration isolation technique using piezoelectric polyvinylidene fluoride (PVDF) polymer was proposed and evaluated. Mathematical equations of the PVDF isolation system were formulated and solved to predict theoretical isolation at variable feedback gains. A physical model made of the PVDF polymer was also designed and tested. Studies conducted by varying the feedback gains and keeping a constant base excitation show that the active piezoelectric isolation is a linear function of feedback voltages at a given frequency and a quadratic function of frequency at constant voltage. The analytical solutions compared favorably with the experimental results.

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